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The Invariant Eigen-Operator Method for *N* Harmonically Coupled Identical Oscillators

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Abstract In this paper, we apply the method of "invariant eigen-operator" to study the Hamiltonian of arbitrary number of coupled identical oscillators and derive their invariant eigen-operator. The results show that, (1) for the system of arbitrary number of identical harmonic oscillators by coordinate coupling or momentum coupling, the invariant eigen operator $\hat{Q}$ of system always has the form of $\hat{Q}=\sum_j g_j\hat{x}_j$ or $\hat{Q}=\sum_j \lambda_j\hat{p}_j$; (2) the energy level gap of the system has two kinds of possibilities: one is that gap only related to ω that the frequency of oscillators; another one is that gap not only related to ω that the frequency of oscillators, but also related to the number of the coupling oscillators.

Keywords Heisenberg equation · Invariant eigen-operator method · Coupled harmonic oscillators

1 Introduction

Quantum theory is one of the most important physical theories of the 20th century. Quantum mechanic has not only advanced our understanding of nature in a profound way but it has also provided the basis on an undoubtedly and indispensably guided principle lying behind contemporary science and technology. Quantum mechanics has, more significantly, changed our view of microscopic world in a way to a completely surprising and unprecedented depth.

In the quantum mechanics, many problems in molecular, atomic, nuclear physics, and quantum optics rely upon the solution of Schrödinger equations for coupled harmonic oscillators. In general, solving various stationary Schrödinger equations leads to eigenvalues and eigenvectors of dynamic Hamiltonians [1, 2]. Fan create the technique of integration within an ordered product (IWOP) of operators, which is definite in physical aim and concise in mathematical description, esphysical that of normal-ordered operators, and try to solve some questions suspended for many years by virtue of it [3, 4]. Until now, it has done an important

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effect in quanta optics and quanta informatics area. IWOP technique work in dynamics in coupled oscillators also has a original charm.

Though the Heisenberg equation stands on the same footing as the Schrödinger equation, it is seldom employed for the purpose of directly deriving energy eigenvalues. Heisenberg does not focus on coordinate or momentum of particle, but on the capacity which can be observed in physics-spectrum, it is corresponding with transition between ability classes. In a very recent paper [5] author has introduced a new method, i.e. the "invariant eigen-operator" (IEO) method to explore energy-level gap of dynamic Hamiltonians.

Follow by the development of quantum informatics and multibody physics, people has built the models of arbitrary number of coupled identical oscillators, and to study its dynamics evolution. In Refs. [6, 7], the solution for an arbitrary number of coupled identical oscillators has been found through a unitary transformation approach. Author present a coordinate representation of the unitary operator that can diagonalize the Hamiltonian. But in terms of only focusing on that system can register the angle, it is still too complex. And then, in this paper, we apply the method of "invariant eigen-operator" to obtain energy gaps of arbitrary number of coupled identical oscillators.

2 The Invariant Eigen-Operator Method

To solve quantum mechanical dynamic problems one sets up Schrödinger equation $i\hbar(d/dt)\psi = \hat{H}\psi$. When Hamiltonian $\hat{H}$ does not contain time explicitly, the solution of stationary Schrödinger equation $iH\Psi_n = E_n\Psi_n$, can straightforwardly depict system energy values and eigenstates. However, owing to the Heisenberg equation of motion [8],

$$i\hbar\frac{d}{dt}\hat{Q} = [\hat{Q}, \hat{H}] \tag{1}$$

which is of the same form and importance as the Schrödinger equation. Since (1) does not involve wavefunctions or eigenvectors, it can hardly be straightforwardly employed to derive energy-level formulas. In Ref. [9], the authors reported that the Heisenberg equation of motion can also be used to deduce the energy-level gap of certain dynamic systems if one can find some appropriate eigen-operators $\hat{Q}$ of the square of the Schrödinger operator $i\hbar(d/dt)$. Its main idea is as follows. For the certain quantum system, there is $\hat{Q}$ satisfying the following eigenvector-like equation

$$\left(i\hbar\frac{d}{dt}\right)^n \hat{Q} = [\cdots[\hat{Q}, \hat{H}], \hat{H}]\cdots] = G^n\hat{Q} \tag{2}$$

where G is real. We can judge that $\sqrt[n]{G}$ is the gap between two adjacent energy levels of the dynamic Hamiltonian $\hat{H}$.

3 The Invariant Eigen-Operator Method for Some Hamiltonians

We talk about the oscillators system of coordinate coupling first. The Hamiltonian of arbitrary number of coupled identical oscillators by coordinate coupling is written as

$$\hat{H}_1 = \sum_{j=1}^{N}\left(\frac{1}{2m}\hat{p}_j^2 + \frac{1}{2}m\omega^2\hat{x}_j^2\right) + \frac{\alpha}{4}\sum_{j,k}^{N}\left(\hat{x}_j - \hat{x}_k\right)^2 \tag{3}$$

According to the invariant eigen-operator method, firstly we should consider the basic commutative relations using the Heisenberg equation ($\hbar = 1$).

$$[\hat{x}_j, \hat{H}] = i\frac{1}{m}\hat{p}_j \tag{4}$$

$$[\hat{p}_j, \hat{H}] = -i\left[(m\omega^2 + N\alpha)\hat{x}_j - \alpha\sum_k^N \hat{x}_k\right] \tag{5}$$

Based on Hamiltonian $\hat{H}_1$ in (3), we suppose that the invariant eigen-operator in the case is

$$\hat{Q}_1 = \sum_{j=1}^N g_j\hat{x}_j \tag{6}$$

The parameter g_js is to be determined.

Substituting (6) into (2), combing (4) and (5), we have

$$[[\hat{Q}_1, \hat{H}], \hat{H}] = \sum_{j=1}^N g_j\left[\left(\omega^2 + N\frac{\alpha}{m}\right)\hat{x}_j - \frac{\alpha}{m}\sum_{k=1}^N \hat{x}_k\right] \tag{7}$$

If the operator $\hat{Q}_1$ satisfies (2), comparing (7) and (6), we obtain a system of linear homogeneous equations,

$$\begin{pmatrix} a+(N-1)b & -b & \dots & -b & -b \\ -b & a+(N-1)b & \dots & -b & -b \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -b & -b & \dots & a+(N-1)b & -b \\ -b & -b & \dots & -b & a+(N-1)b \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{N-1} \\ g_N \end{pmatrix} = G_1\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{N-1} \\ g_N \end{pmatrix} \tag{8}$$

where

$$a = \omega^2, \qquad b = \frac{2\alpha}{m}$$

Equations (8) eigenvalues are: ω^2, $\omega^2 + N\frac{\alpha}{m}$. Thus the energy-level gap of Hamiltonian $\hat{H}_1$ also has two kinds of probabilities.

$$G_{11} = \omega \quad \text{or} \quad G_{12} = \sqrt{\omega^2 + N\frac{\alpha}{m}} \tag{9}$$

What should be pointed out is that: the eigenvalue G_{12} of the homogeneous linear equation group is actually the multiple roots of $N - 1$. It tells us that: for the arbitrary number of coupled identical oscillators is described by (3), there are only two kinds of frequency which after decoupling.

$$\Omega_1 = G_{11}, \qquad \Omega_2 = G_{12} \tag{10}$$

The energy eigen-value of the system reads

$$E = n_1 \hbar \Omega_1 + \hbar \Omega_2 \sum_{j=2}^{N} n_j$$

In order to be able to understand intuitively the method of "invariant eigen-operator", for the coupled quantum system of two harmonic oscillators, we discussed a simple case.

The Hamiltonian of two coupled harmonic oscillators is

$$H_{\text{two}} = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2} m \omega_0^2 (x_1^2 + x_2^2) - \lambda x_1 x_2$$

The H_{two} can be rewritten

$$H_{\text{two}} = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2} m \left(\omega_0^2 - \frac{\lambda}{m}\right)(x_1^2 + x_2^2) - \frac{\lambda}{4} \sum_{j,k}^{2} (x_j - x_k)^2$$

If $N = 2$, $\omega^2 = \omega_0^2 - \lambda/m$, $\alpha = -\lambda$, from (9) and (10), we can obtain

$$\Omega_{\text{two}-1} = \sqrt{\omega^2 - \frac{\lambda}{m}}, \qquad \Omega_{\text{two}-2} = \sqrt{\omega^2 + \frac{\lambda}{m}}$$

It is completely the same as the result of Ref. [2].

The Hamiltonian of four coupled harmonic oscillators is

$$\hat{H}_{\text{four}} = \sum_{j=1}^{4} \left(\frac{1}{2m} \hat{p}_j^2 + \frac{1}{2} m \omega^2 \hat{x}_j^2 \right) + \frac{k}{4} \sum_{j,k}^{4} (\hat{x}_j - \hat{x}_k)^2$$

If $N = 4$, $\alpha = k$, from (9) and (10), we can obtain

$$\Omega_{\text{four}-1} = \omega, \qquad \Omega_{\text{four}-2} = \sqrt{\omega^2 + \frac{4k}{m}}$$

It is completely the same as the result of Ref. [10]. Comparing with the way used by the Refs. [2, 10], the "invariant eigen-operator" (IEO) method is much simple.

In fact, for the quantum system describing by Hamiltonian (3), we can find that another invariant eigen-operator $\hat{Q}_1'$ is

$$\hat{Q}_1' = \sum_{j=1}^{N} g_j' \hat{p}_j \tag{11}$$

Here g_1' is to be determined. Substituting (11) into (2), combing (4) and (5), we have

$$[[\hat{Q}_1', \hat{H}], \hat{H}] = \sum_{j=1}^{N} g_j' \left[\left(\omega^2 + N \frac{\alpha}{m} \right) \hat{p}_j - \frac{\alpha}{m} \sum_{k=1}^{N} \hat{p}_k \right] \tag{12}$$

If the operator $\hat{Q}'_1$ satisfies (2), comparing (12) and (11), we obtain a system of linear homogeneous equations. This equation group is similar with (8).

$$\begin{pmatrix} a+(N-1)b & -b & \dots & -b & -b \\ -b & a+(N-1)b & \dots & -b & -b \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -b & -b & \dots & a+(N-1)b & -b \\ -b & -b & \dots & -b & a+(N-1)b \end{pmatrix} \begin{pmatrix} g'_1 \\ g'_2 \\ \vdots \\ g'_{N-1} \\ g'_N \end{pmatrix} = G_1 \begin{pmatrix} g'_1 \\ g'_2 \\ \vdots \\ g'_{N-1} \\ g'_N \end{pmatrix} \tag{13}$$

Working out the eigenvalues of (13) can also get the energy level gap giving by (3).

Follow by the development of quanta informatics and multibody physics, people has built the oscillators models of momentum coupling, and studying its dynamics evolution is also an important task of quanta theory.

The Hamiltonian of arbitrary number of coupled identical oscillators by momentum coupling is written as

$$\hat{H}_2 = \sum_{j=1}^{N} \left(\frac{1}{2m} \hat{p}_j^2 + \frac{1}{2} m\omega^2 \hat{x}_j^2 \right) + \frac{\beta}{4} \sum_{j,k}^{N} (\hat{p}_j - \hat{p}_k)^2 \tag{14}$$

According to the invariant eigen-operator method, firstly we should consider the basic commutative relations using the Heisenberg equation

$$[\hat{x}_j, \hat{H}] = i \left[\frac{1}{m} \hat{p}_j + N\beta \hat{p}_j - \beta \sum_{k}^{N} \hat{p}_k \right] \tag{15}$$

$$[\hat{p}_j, \hat{H}] = -im\omega^2 \hat{x}_j \tag{16}$$

Based on Hamiltonian $\hat{H}_2$ in (14), we suppose that the invariant eigen-operator in the case is

$$\hat{Q}_2 = \sum_{j=1}^{N} h_j \hat{p}_j \tag{17}$$

The parameter h_js is to be determined.

Substituting (17) into (2), combing (15) and (16), we have

$$[[\hat{Q}_2, \hat{H}], \hat{H}] = \omega^2 \sum_{j=1}^{N} h_j \left[(1 + m\beta N) \hat{x}_j - m\beta \sum_{k}^{N} \hat{x}_k \right] \tag{18}$$

If the operator $\hat{Q}_2$ satisfies (2), comparing (18) and (17), we obtain a system of linear homogeneous equations,

$$\begin{pmatrix} a'+(N-1)b' & -b' & \dots & -b' & -b' \\ -b' & a+(N-1)b' & \dots & -b' & -b' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -b' & -b' & \dots & a'+(N-1)b' & -b' \\ -b' & -b' & \dots & -b' & a'+(N-1)b' \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{N-1} \\ h_N \end{pmatrix}$$
$$= G_2 \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{N-1} \\ h_N \end{pmatrix} \tag{19}$$

where

$$a' = \omega^2, \qquad b' = m\beta\omega^2$$

Equations (19) eigenvalues are: $\omega^2, \omega^2(1+m\beta N)$. Thus the energy-level gap of Hamiltonian $\hat{H}_2$ also has two kinds of probabilities.

$$G_{21} = \omega \quad \text{or} \quad G_{22} = \sqrt{\omega^2(1+m\beta N)} \tag{20}$$

Obviously, the eigenvalue G_{22} of the homogeneous linear equation group is actually the multiple roots of $N-1$. It tells us that: for the arbitrary number of coupled identical oscillators is described by (14), there are only two kinds of frequency which after decoupling.

In Ref. [11], the Hamiltonian of two coupled harmonic oscillators was studied

$$H'_{\text{two}} = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}m\omega_0^2(x_1^2 + x_2^2) - k_1 p_1 p_2$$

The H'_{two} can be rewritten

$$H'_{\text{two}} = \frac{1}{2m}(1+mk_1)(p_1^2 + p_2^2) + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) + \frac{k_1}{4}\sum_{j,k}^{2}(p_j - p_k)^2$$

If $N = 2$, $\beta = k_1$, from (20), we can obtain

$$\Omega'_{\text{two}-1} = \sqrt{\omega^2(1-mk_1)}, \qquad \Omega'_{\text{two}-2} = \sqrt{\omega^2(1+mk_1)}.$$

It is completely the same as the result of Ref. [11].

If the quantum system describing by Hamiltonian (14), we can also find that another invariant eigen-operator $\hat{Q}'_2$ is

$$\hat{Q}'_2 = \sum_{j=1}^{N} h'_j \hat{p}_j \tag{21}$$

Though $\hat{Q}'_2$, the process that is similar with above, can also give out the energy level gap of system.

4 Conclusion

We have adopted the invariant eigen-operator method to tackle some special coupled oscillator model. This approach seems concise and direct and can be extended to tackle other Hamiltonian models. From above results, we can easily know that: for the quantum system of coupled harmonic oscillators by coordinate coupling, momentum coupling or both coordinate coupling and momentum coupling, the form of invariant eigen-operator $\hat{Q}$ is $\hat{Q} = \sum_j g_j \hat{x}_j$ or $\hat{Q} = \sum_j h_j \hat{p}_j$. Coefficient g_js or h_js is determined by both the inherent parameters of coupled harmonic oscillators and coupling parameter. For the ability level of the arbitrary number of coupled identical oscillators system which has both of the coordinate coupling and the momentum coupling, using the way above, it can also get the result easily.

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References

1. Estes, L.E.: Quantum mechanical description of two coupled harmonic oscillators. Phys. Rev. **175**, 286 (1968)
2. Carusotto, S.: Theory of a quantum anharmonic oscillator. Phys. Rev. A **38**, 3249 (1988)
3. Louisell, W.H.: Quantum Statistical Properties of Radiation. Wiley, New York (1973)
4. Wick, G.: The evaluation of the collision matrix. Phys. Rev. **80**, 268 (1950)
5. Fan, H.Y., Li, C.: Invariant eigen-operator of the square of Schrodinger operator for deriving energy-level gap. Phys. Lett. A **321**, 75 (2004)
6. Fan, H.Y.: Unitary operator for an arbitrary number of coupled identical oscillators. Phys. Rev. A **47**, 2379 (1992)
7. Michelot, F.: Solution for an arbitrary number of coupled identical oscillators. Phys. Rev. A **45**, 4271 (1992)
8. Fan, H.Y., Tang, X.B., Hu, H.P.: Deriving energy-gap of some nonlinear Hamiltonians by invariant eigen-operator method. Commun. Theor. Phys. **50**, 674 (2008)
9. Fan, H.Y., Wu, H.: Vibrational spectrum for the linear lattice chain gained by of the "Invariant eigen-operator" method. Int. J. Mod. Phys. B **19**, 4073 (2005)
10. Fan, H.Y.: Unitary transformation for four Harmonically coupled identical oscillators. Phys. Rev. A **42**, 4377 (1990)
11. Liu, Y.M., Ji, Y.H.: The invariant eigen-operator method for Hamiltonians with coordinates—coordinates coupling terms. Int. J. Theor. Phys. (2009, accepted)